Math 8 Homework 2

1 Set Basics

For each of the following, determine if the statement is true or false. In each case give a reason or a counterexample.

(a) $A \cup B \subseteq A \cap B$ for any sets A and B.

(b) $\varnothing \in \mathcal{P}(A)$ for any set A.

(c) $\{\varnothing\} \in \mathcal{P}(\{\varnothing, \{\varnothing\}\})$

(d) $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$ for any sets A and B.

(e) $\mathcal{P}(A) \times \mathcal{P}(B) = \mathcal{P}(A \times B)$ for any sets A and B.

2 Set Proofs

Let A, B, C and D be arbitrary sets.

(a) Prove that $A \subseteq \varnothing$ if and only if $A = \varnothing$.

(b) Prove that $A \cup B = A$ if and only if $B \subseteq A$.

(c) Prove that if $A \neq \varnothing$ and $A \times B = \varnothing$, then $B = \varnothing$.

(d) Suppose that $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$. Prove that either $A \subseteq B$ or $A \cap B = \varnothing$.

(e) Suppose that $A \cup B \subseteq C \cup D$ and $A \cap B = \varnothing$. If $C \subseteq A$, prove that $B \subseteq D$.

(f) Let $A \Delta B = (A - B) \cup (B - A)$. Prove that $A \Delta \varnothing = A$.

(g) Prove that $A \Delta B = \varnothing$ if and only if $A = B$.

(h) Prove or disprove: $A\Delta(B\Delta C) = (A\Delta B)\Delta C$.

3 Indexed Collections

(a) Given an indexed family of sets $\{A_i : i \in I\}$ and a set B, consider the statement

$$B - \left(\bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} (B - A_i).$$

If this is generally true, prove it. If it could be false, give a counterexample.

(b) Let f be a real-valued function on $\mathbb{R}$. Prove that

$$\bigcup_{n=1}^{\infty} \{x \in \mathbb{R} : f(x) \geq 1/n\} = \{x \in \mathbb{R} : f(x) > 0\}.$$

(c) Given a sequence of sets $A_1, A_2, A_3, \ldots$ define two new sets

$$I = \bigcup_{n=1}^{\infty} \left(\bigcap_{m=n}^{\infty} A_m \right)$$

$$S = \bigcap_{n=1}^{\infty} \left(\bigcup_{m=n}^{\infty} A_m \right)$$

Prove that $I \subseteq S$.

(d) With I and S as above, give an example of sets (A_n) for which $I \neq S$.